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HALF-MOUFANG GENERALIZED HEXAGONS

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ABSTRACT

If Γ is a half-Moufang generalized hexagon, then Γ is Moufang. We also give a very short proof that a generalized hexagon admitting a split BN-pair is a Moufang hexagon.

1. Introduction

In classifying groups acting on geometries, in particular groups with BN-pairs (see Section 3), the Moufang condition is in most cases the crucial property which allows one to identify the groups using the classification of the Moufang polygons by Tits and Weiss [11]. However, often only part of the Moufang condition can be verified and hence it is very useful to know that this part is already sufficient.

For finite generalized quadrangles it was known that the half-Moufang property implies the Moufang property [3]. This result was generalized to arbitrary quadrangles in [7]. For generalized hexagons, [4] shows that a half-Moufang hexagon where all given elations are central elations satisfies the Moufang condition. We here give a new and direct proof of this result (except for the smallest case $G_2(2)$ for which there is already another direct proof due to Van Maldeghem [12] 6.3.2, case $t = 2$) and generalize it to prove the result stated in the abstract. Note that this is a real generalization even for the smallest case since we do not assume that the given elations are central.

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As a by-product Theorem 4.2 gives a very short proof of the fact that if a group with a split BN-pair acts on a generalized hexagon, it has to satisfy the Moufang property (see [9]).

The general half-Moufang case is open for generalized n-gons with $n = 2m \geq 8$.

2. Set-up

A generalized n-gon Γ is a bipartite graph with valencies at least 3, diameter n and girth $2n$. For generalized hexagons we have $n = 6$. The vertices of this graph are called the **elements** of Γ. The set of elements adjacent to some element $x \in \Gamma$ is denoted by $\Gamma_1(x)$, and more generally $\Gamma_i(x)$ denotes the set of elements of (graph theoretic) distance i from x.

We say that two elements have the same type if they belong to the same class in the bipartition of Γ. Similarly, we say that two paths have the same type if they have the same length and start or end with elements of the same type.

If $G \leq \mathrm{Aut}(\Gamma)$, we denote by $G_{x_0}^{[i]}$ the subgroup of G fixing all elements of $\Gamma_i(x_0)$ (and then it automatically fixes all sets $\Gamma_j(x_0)$ pointwise, for $0 \leq j \leq i$). Further, for elements $x_1, \ldots, x_k$, we set $G_{x_0,x_1,\ldots,x_k}^{[i]} = G_{x_0}^{[i]} \cap G_{x_1}^{[i]} \cap \cdots \cap G_{x_k}^{[i]}$. For $i = 0$, we usually omit the superscript [0]. For every simple path $(x_0, \ldots, x_{n+1})$ of length $n+1$ and every i with $0 \leq i \leq n$, we have $G_{x_0,\ldots,x_{n+1}} \cap G_{x_i,x_{i+1}}^{[1]} = 1$.

Let $(x_0, \ldots, x_n)$ be a simple path. An **elation** (or $(x_0,\ldots,x_n)$**-elation**) g of Γ is a member of $G_{x_1,\ldots,x_{n-1}}^{[1]}$. If n is even, an elation $g \in G_x^{[n/2]}$ is called **central** and x is called the **center** of the elation. The group $G_{x_1,\ldots,x_{n-1}}^{[1]}$ of elations acts freely on $\Gamma_1(x_0) \setminus \{x_1\}$ and on $\Gamma_1(x_n) \setminus \{x_{n-1}\}$. If this action is transitive, then we say that the path $(x_1, x_2, \ldots, x_{n-1})$ is a **Moufang path**. If all simple $(n-2)$-paths are Moufang, then we say that Γ is a **Moufang polygon**. If n is even, and if all simple paths of length $n-2$ starting with an element of fixed type are Moufang, then we say that Γ is **half-Moufang** for that type. If all elations of that type are central elations, then we say that Γ is **half-Moufang with all central elations**. Notice that the half-Moufang condition is the same as the Moufang condition for generalized n-gons where n is odd. It is well-known that Moufang n-gons exist only for $n = 3, 4, 6, 8$. Background on the Moufang condition for generalized n-gons can be found in [12] and in the full classification [11].

Notation: Regarding commutators and conjugation, we use the notation $g^h = h^{-1}gh$. We also write $g^{-h} = (g^{-1})^h$ and $[g,h] = g^{-1}h^{-1}gh = h^{-g}h = g^{-1}g^h$. We let automorphisms act on the right, so we use exponential notation.

If H is a group acting on a set Ω and $A \subseteq \Omega$, we let H_A denote the pointwise and $H_{\{A\}}$ denote the setwise stabilizer of A in H.

The following well-known observations are at the heart of many arguments.

2.1 LEMMA: *Let H be a group acting on a set Ω. Let $A \subseteq \Omega$. Suppose that $g \in H_A$ and let $h \in H$. Then $[g,h] \in H_A$ if and only if $g \in H_{A^h}$.*

For $G \leq \mathrm{Aut}(\Gamma)$ we have in particular: if $h \in G_x$ then $[g,h] \in G_x^{[i]}$ if and only if $g \in G_x^{[i]}$.

Notice also that $[g,h] = [g,k]$ implies that hk^{-1} centralizes g and hence $[g, hk^{-1}] = 1$.

3. BN-pairs

Generalized polygons were introduced by Tits [10]. The standard examples arise from groups with an irreducible spherical BN-pair of rank 2, hence in particular from groups of Lie type. For the purpose of the present paper, the following geometric definition of such a BN-pair will do.

Let Γ be a generalized n-gon, and let G be a group acting (not necessarily effectively) on Γ such that each element of G acts as a type preserving graph automorphism. If G acts transitively on the set of ordered $2n$-cycles of Γ, then we say that G is a group with an **irreducible spherical BN-pair of rank** 2, or briefly, with a **BN-pair**. We say that Γ admits a BN-pair if $G = \mathrm{Aut}(\Gamma)$ has a BN-pair. The subgroups B and N of G forming this BN-pair can be described as follows. Let $A = (x_0, x_1, \ldots, x_{2n-1}, x_{2n} = x_0)$ be a $2n$-cycle in Γ. Let $B = G_{x_0,x_1}$ and $N = G_{\{A\}}$. Then $T = B \cap N$ fixes A pointwise, and clearly $T \lhd N$. By the transitivity of G on ordered $2n$-cycles, we see that the **Weyl group** $W = N/T$ is the dihedral group of order $2n$. The BN-pair is called **split** if there is a normal nilpotent subgroup $U \lhd B$ with $B = U(B \cap N)$ or, equivalently, with U acting transitively on the $2n$-cycles containing (x_0, x_1). *Finite* split BN-pairs of rank 2 were classified by Fong and Seitz [2] and the general classification was obtained in [8, 9, 6]. Groups with a split BN-pair of rank 2 are those associated with the group of k-rational points of an absolutely simple algebraic group of relative rank 2.

From now on, throughout the paper we let Γ denote a generalized hexagon and we fix a path $(x_0, \ldots, x_6)$. Unless explicitly stated otherwise we assume that Γ is half-Moufang for paths of the same type as $(x_1, \ldots, x_5)$. In particular, the group $U = G_{x_1,\ldots,x_5}^{[1]}$ acts transitively on $\Gamma_1(x_0) \setminus \{x_1\}$. Let $G \leq \mathrm{Aut}(\Gamma)$ denote the group generated by all elations of this type.

3.1 LEMMA: *Let* $x_{-1} \in \Gamma_1(x_0)\setminus\{x_1\}$. *If* $|\Gamma_1(x_0)| \geq 4$, *then* $V = G^{[1]}_{x_1} \cap G_{x_{-1},\ldots,x_5}$ *acts transitively on* $\Gamma_1(x_5)\setminus\{x_4\}$. *If* $|\Gamma_1(x_0)| = 3$, *then* $G^{[1]}_{x_0,x_1,x_2}$ *acts transitively on* $\Gamma_2(x_3) \setminus \Gamma_1(x_2)$.

Proof: First assume $|\Gamma_1(x_0)| = 3$. Let $\gamma = (x_2, x_3, x_4', x_5', x_6', x_7')$ be a simple path with $x_4 \neq x_4'$. Complete γ into a closed 12-cycle

$$(x_7', x_6', x_5', x_4', x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11} = x_7').$$

Let p_0 denote the projection of x_9 onto x_2 (i.e., p_0 is the unique element in $\Gamma_1(x_2)$ with $d(x_9, p_0) = 4$), and let $q \in \Gamma_1(x_2) \setminus \{x_3, p_0\}$. Then $d(x_8, x_2) = d(x_{10}, x_2) = 6$. Let γ_1 denote the unique 6-path $(x_8, \ldots, q, x_2)$ and γ_2 the unique 6-path $(x_{10}, \ldots, q, x_2)$. Let α_1 denote the γ_1-elation with $x_7^{\alpha_1} = x_9$ (and so $x_3^{\alpha_1} = p_0$), let α_2 denote the γ_2-elation with $x_9^{\alpha_2} = x_7'$ (and hence $p_0^{\alpha_2} = x_3$). Then $g = \alpha_1\alpha_2 \in G^{[1]}_{q,x_2}$ and $x_5^g = x_5'$. Let β be the $(x_2, \ldots, x_8)$-elation with $q^\beta = x_1$. Then $g^\beta \in G^{[1]}_{x_1,x_2}$ and $x_4^{g^\beta} = x_4'$. Let β_1 be some $(x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, x_4')$-elation with $x_5^{g^\beta\beta'} = x_5'$ and let β_2 be the $(x_0, x_1, x_2, x_3, x_4', x_5', x_6')$-elation with $x_{-1}^{g^\beta\beta_1\beta_2} = x_{-1}$. Then $g^\beta\beta_1\beta_2 \in G^{[1]}_{x_0,x_1,x_2}$ is as desired.

Now assume that $|\Gamma_1(x_0)| \geq 4$. We repeat the first part of the previous argument twice. Let $y_6 \in \Gamma_1(x_5) \setminus \{x_4, x_6\}$. Let $\gamma = (x_2, x_3, x_4', x_5', x_6', x_7')$ be a simple path with $x_4 \neq x_4'$. Complete γ in two distinct ways into 12-cycles

$$(x_7', x_6', x_5', x_4', x_3, x_4, x_5, x_6, \ldots, x_{10}, x_{11} = x_7')$$

and

$$(x_7', x_6', x_5', x_4', x_3, x_4, x_5, y_6, \ldots, y_{10}, y_{11} = x_7').$$

Let p_0 and p_1 denote the respective projections of x_9 and y_9 onto x_2, and let $q \in \Gamma_1(x_2)\setminus\{x_3, p_0, p_1\}$. Then $d(x_8, x_2) = d(y_8, x_2) = d(x_{10}, x_2) = d(y_{10}, x_2) = 6$. Let $\gamma_1 = (x_8, \ldots, q, x_2)$ be a path of length 6 and let α_1 denote the γ_1-elation with $x_7^{\alpha_1} = x_9$ (and so $x_3^{\alpha_1} = p_0$); let $\gamma_2 = (x_{10}, \ldots, q, x_2)$ be a path of length 6, and α_2 denote the γ_2-elation with $x_9^{\alpha_2} = x_7'$ (and hence $p_0^{\alpha_2} = x_3$). Let $\gamma_3 = (y_{10}, \ldots, q, x_2)$ be a path of length 6 and α_3 denote the γ_3-elation with $x_7'^{\alpha_3} = y_9$, so $x_3^{\alpha_3} = p_1$. Finally, let $\gamma_4 = (y_8, \ldots, q, x_2)$ be a path of length 6 and let α_4 denote the γ_4-elation with $y_9^{\alpha_4} = y_7$, and so $p_1^{\alpha_4} = x_3$.

Then $g = \alpha_1\alpha_2\alpha_3\alpha_4 \in G^{[1]}_q$ and $x_6^g = y_6$. Let β be the $(x_2, \ldots, x_8)$-elation with $q^\beta = x_1$. Let α_0 be the $(x_0^{\beta^{-1}}, q, x_2, \ldots, x_6)$-elation with $x_{-1}^{\beta^{-1}g\alpha_0} = x_{-1}^{\beta^{-1}}$. Then $h = (g\alpha_0)^\beta \in G^{[1]}_{x_1} \cap G_{x_{-1},\ldots,x_5}$ and $x_6^h = x_6'$, showing $V = G^{[1]}_{x_1} \cap G_{x_{-1},\ldots,x_5}$ to act transitively on $\Gamma_1(x_5) \setminus \{x_4\}$. ∎

3.2 COROLLARY: *If $|\Gamma_1(x_0)| \geq 4$, then G has a BN-pair. For $|\Gamma_1(x_0)| = 3$, G acts transitively on all paths of length 5 starting with the same type of element.*

Proof: This follows immediately from the previous lemma. ∎

3.3 Definition: We say that the root action for x_0 is **independent of the root** if for each path $(x_0, x_1, x_2', x_3', x_4', x_5', x_6')$ the following holds: let $U = G^{[1]}_{x_1,x_2,x_3,x_4,x_5}$ and $U_1 = G^{[1]}_{x_1,x_2',x_3',x_4',x_5'}$, and let $\Omega = \Gamma_1(x_0) \setminus \{x_1\}$. Then $U_1|_\Omega = U|_\Omega$.

3.4 COROLLARY: *Assume $|\Gamma_1(x_0)| \geq 4$. If the root action for x_0 is independent of the root, then $V = G^{[1]}_{x_0,x_1,x_2} \cap G_{x_4,x_5}$ acts regularly on $\Gamma_1(x_5) \setminus \{x_4\}$ and on $\Gamma_1(x_{-1}) \setminus \{x_0\}$.*

Proof: We keep the notation of the proof of Lemma 3.1. If the root action for x_0 is independent of the root, then by construction we have $\alpha_1\alpha_2|_{\Gamma_1(x_2)} = \alpha_3\alpha_4|_{\Gamma_1(x_2)} = id|_{\Gamma_1(x_2)}$. By choice of α_0, we also have $\alpha_1 \cdots \alpha_4\alpha_0|_{\Gamma_1(x_0^{\beta^{-1}})} = id|_{\Gamma_1(x_0^{\beta^{-1}})}$ and hence $h = (\alpha_1 \cdots \alpha_4\alpha_0)^\beta \in G^{[1]}_{x_0,x_1,x_2} \cap G_{x_4,x_5}$. The regularity follows from the fact that $G^{[1]}_{x_0,x_1,x_2} \cap G_{x_4,x_5,x_6} = 1$. ∎

In the following proposition we do not assume that Γ is half-Moufang:

3.5 PROPOSITION: *Let Γ be a generalized hexagon with $G = Aut(\Gamma)$ transitive on the set of paths of length 2 of the same type. Assume that $\beta \in G^{[2]}_{x_0,x_2} \setminus \{1\}$, and that $G^{[1]}_{x_0} \cap G_{x_2}$ is transitive on $\Gamma_2(x_2) \setminus \Gamma_1(x_1)$. Then Γ is half-Moufang and all elations are conjugate to β. If $\beta \in G^{[2]}_{x_0,x_1,x_2}$, it suffices that $G^{[1]}_{x_0}$ is transitive on $\Gamma_1(x_2) \setminus \{x_1\}$.*

Proof: Let (x_{-2}, x_{-1}, x_0) be an extension of $(x_0, \ldots, x_6)$. We will show that $G^{[2]}_{x_0,x_{-2}}$ is transitive on $\Gamma_1(x_2) \setminus \{x_1\}$. Let $p \in \Gamma_1(x_2) \setminus \{x_1, x_3\}$ and let $\alpha \in G^{[1]}_{x_4}$ be such that $p = x_1^\alpha$. Then $[\alpha, \beta] \in G^{[2]}_{x_2} \cap G^{[1]}_{x_4}$. Let $y = x_{-2}^{\beta^{-\alpha}}$, and let $g \in G^{[1]}_y$ be such that $x_2^g = x_{-2}$. (It suffices to choose $g \in G^{[1]}_y$ with $x_1^g = x_{-1}$ if $\beta \in G^{[2]}_{x_0,x_1,x_2}$.) Put $\gamma = \beta^g$, so $\gamma \in G^{[2]}_{x_0,x_{-2}}$, and $[\beta, g] = \beta^{-1}\gamma \in G^{[1]}_y$. We claim that $[\alpha,\beta]\beta^{\alpha\gamma} \in G^{[2]}_{x_2} \cap G^{[1]}_{x_{-2}} = 1$. Clearly, $[\alpha,\beta]\beta^{\alpha\gamma} \in G^{[2]}_{x_2}$. Now, $\beta\gamma^{-1} \in G^{[1]}_y$. Hence by construction $(\beta\gamma^{-1})^{\beta^\alpha} \in G^{[1]}_{x_{-2}}$. Since $\gamma \in G^{[1]}_{x_{-2}}$, we have $(\beta\gamma^{-1})^{\beta^\alpha}\gamma = [\alpha,\beta]\beta^{\alpha\gamma} \in G^{[1]}_{x_{-2}}$, as claimed.

Thus, $[\alpha, \beta] = \beta^{-\alpha\gamma}$ is a non-trivial elation in $G^{[2]}_{x_2,x_0^{\alpha\gamma}}$. Since $[\alpha,\beta] \in G^{[1]}_{x_4}$, we must have $x_1^{\alpha\gamma} = p^\gamma = x_3$. Since $p \in \Gamma_1(x_2)$ was arbitrary, we have proved the claim. ∎

Suppose again that Γ is half-Moufang for $(x_1, \ldots, x_5)$.

3.6 LEMMA: *If* $U_0 = G^{[2]}_{x_0,x_2}, U_1 = G^{[2]}_{x_2,x_4}, U_2 = G^{[2]}_{x_4,x_6}$, *we have* $[U_0, U_1] = 1$ *and* $[U_0, U_2] = U_1$.

Proof: Note that we do not assume that $G^{[2]}_{x_2,x_4}$ is transitive on $\Gamma_1(x_0) \setminus \{x_1\}$! By Lemma 2.1 we have $[U_0, U_1] \leq G^{[2]}_{x_0,x_2,x_4} = 1$. Similarly, $[U_0, U_2] \leq U_1$. To see that equality holds, let $\alpha \in U_0$, and let $(x_0, \ldots, x_6, x_7, \ldots, x_{12} = x_0)$ be a closed cycle. Let $\gamma \in G^{[1]}_{x_5,x_6,x_7,x_8,x_9}$ with $x_5^{\alpha\gamma} = x_3$, and hence $x_6^{\alpha\gamma} = x_2$. Then for any $\beta \in U_2$ we have $\beta^{\alpha\gamma} \in U_1$. Furthermore, $[\beta^{-\alpha}, \gamma] = \beta^{\alpha}\beta^{-\alpha\gamma} \in G^{[1]}_{x_6}$. Since $\beta \in G^{[1]}_{x_6}$, we thus have $\beta^{-1}\beta^{\alpha}\beta^{-\alpha\gamma} = [\beta, \alpha]\beta^{-\alpha\gamma} \in U_1 \cap G^{[1]}_{x_6} = 1$. So $[\beta, \alpha] = \beta^{\alpha\gamma}$ for any $\beta \in U_2$. Thus if $\delta \in U_1$, then $\delta^{(\alpha\gamma)^{-1}} \in U_2$ and $[\delta^{(\alpha\gamma)^{-1}}, \alpha] = \delta$. Hence, $[U_0, U_2] = U_1$. ∎

3.7 LEMMA: *If* $U = G^{[2]}_{x_2,x_4}$, *then* U *is abelian.*

Proof: Let $\alpha \in G^{[2]}_{x_2,x_4}$, and let $y \in \Gamma_2(x_2) \setminus (\Gamma_1(x_1) \cup \Gamma_1(x_3))$. Let δ be an elation for some path $(\ldots, y, p, x_2)$ with $x_3^{\delta} = x_1$. Then $\alpha^{\delta} \in G^{[2]}_{x_4^{\delta},x_2}$ and hence $[\alpha^{\delta}, \beta] = 1$ for all $\beta \in U$ by Lemma 3.6. Since $\delta \in G^{[1]}_y$, we have $[\alpha, \beta] \in G^{[1]}_y$ and hence $[\alpha, \beta] = 1$ for all $\beta \in U$. Thus, U is abelian. ∎

3.8 COROLLARY: *If* $U = G^{[1]}_{x_1,\ldots,x_5} = G^{[2]}_{x_2,x_3,x_4}$, *then the root action for* x_0 *is independent of the root.*

Proof: Let $(x_0, x_1, x_2', x_3', x_4', x_5', x_6')$ be a path, and $U_1 = G^{[2]}_{x_2',x_3',x_4'}$. Then $[U, U_1] = G^{[2]}_{x_2,x_1,x_2'}$ by Lemma 3.6, showing that U and U_1 centralize each other on the set $\Omega = \Gamma_1(x_0) \setminus \{x_1\}$. Since both groups are abelian by Lemma 3.7 and regular on Ω, the claim now follows from the fact that the actions of two regular abelian groups centralizing each other coincide ([1] Thm. 4.2.A). ∎

4. Half-Moufang hexagons

4.1 THEOREM (cf. [4]): *If* Γ *is a half-Moufang generalized hexagon with all central elations, then* Γ *is Moufang.*

Proof: Suppose $U = G^{[3]}_{x_3}$ is transitive on $\Gamma_1(x_0)$. Quoting [4] Sect. 11 or [12] 6.3.2 (case $t = 2$) for the case $|\Gamma_1(x_0)| = 3$, we may assume that $|\Gamma_1(x_0)| \geq 4$.

By Corollary 3.8 and Corollary 3.4 the group $V = G^{[1]}_{x_2,x_3,x_4} \cap G_{x_{-1}}$ acts regularly on $\Gamma_1(x_5)$. We claim that $V \leq G^{[1]}_{x_0,x_2,x_3,x_4}$. Suppose not and let $g \in V \setminus G_q$ for some $q \in \Gamma_1(x_0)$. Let α be an elation with center x_{-1}, and let $\beta \in U$ be such that $x_{-1}^{\beta} = q$. Then $[\alpha,\beta] \in G_{x_1}^{[3]}$ and hence $[g,[\alpha,\beta]] = [g,\alpha^{-1}\alpha^{\beta}] \in G_{x_1}^{[3]} \cap G_{x_4}^{[1]} = 1$ by Lemma 2.1. Since $[g,\alpha] = 1$, we thus also have $[g,\alpha^{\beta}] = 1$. But this is impossible by Lemma 2.1 unless g fixes q.

It is left to show that $V \le G_{x_1}^{[1]}$. To see this let $v \in V$ and $\alpha \in G_{x_{-1}}^{[3]} \setminus \{1\}$. Then $[v,\alpha] = 1$ by Lemma 2.1, showing that $v \in G_{x_3^{\alpha}}^{[1]}$. Now let $\beta \in G_{x_5}^{[3]}$ with $x_1^{\beta} = x_3^{\alpha}$. Then again $[v,\beta] = 1$ by Lemma 2.1, and so $v \in G_{x_1}^{[1]}$. Thus $V \le G_{x_0,x_1,x_2,x_3,x_4}^{[1]}$ consists of elations. ∎

In the following theorem, we do not assume that Γ is half-Moufang.

4.2 Theorem ([9]): *Let G be a group with a split BN-pair of rank 2 acting on a generalized hexagon Γ. Then Γ is Moufang and G contains its little projective group.*

Proof: By [5] Prop. 3.5, $G_{x_0}^{[1]}$ is transitive on $\Gamma_2(x_2) \setminus \Gamma_1(x_1)$ for all $x_0 \in \Gamma$. By [9] Prop. 4.1, either $Z(U)$ consists of central elations, or both $G_{x_0,x_2}^{[2]}$ and $G_{x_1,x_3}^{[2]}$ are nontrivial. In the first case we are done by Proposition 3.5 and Theorem 4.1, in the second case we just use Proposition 3.5 to obtain all elations of both types. ∎

From now on, until the end of the paper, we assume again that Γ is half-Moufang and that $U = G_{x_1,\ldots,x_5}^{[1]}$ acts transitively on $\Gamma_1(x_0) \setminus \{x_1\}$.

4.3 Lemma: *If $G_{x_4}^{[2]} \cap U \ne 1$, then $U = G_{x_2,x_4}^{[2]}$.*

Proof: First we show that if $G_{x_4}^{[2]} \cap U \ne 1$, then also $G_{x_2,x_4}^{[2]} \cap U \ne 1$. So let $\alpha \in G_{x_4}^{[2]} \cap U \setminus \{1\}$, and let $\beta \in G_{x_6}^{[2]} \setminus G_{x_3}$ by Corollary 3.2. Then $[\alpha,\beta] \in G_{x_4,x_6}^{[2]} \setminus \{1\}$.

So let $\alpha \in G_{x_2,x_4}^{[2]}$, let $(x_0,\ldots,x_6,x_7,\ldots,x_{12} = x_0)$ be a closed cycle and let $\beta \in G_{x_5,x_6,x_7,x_8,x_9}^{[1]}$. Then $[\alpha,\beta] \in G_{x_4}^{[2]} \cap G_{x_6}^{[1]}$. Let $\gamma \in G_{x_{11},x_0,x_1,x_2,x_3}^{[1]}$ with $x_2^{\beta^{\gamma}} = x_6$. Then $\alpha^{\beta\gamma} \in G_{x_4,x_6}^{[2]}$ and $[\alpha,\beta]\alpha^{-\beta\gamma} \in G_{x_2}^{[1]}$. By Lemma 3.6, there is some $\delta \in G_{x_6,x_8}^{[2]}$ such that $[\alpha,\delta] = \alpha^{\beta\gamma}$. Hence $h = [\alpha,\delta]^{-1}[\alpha,\beta] = \alpha^{-\delta}\alpha^{\beta} \in G_{x_2,x_6}^{[1]} \cap G_{x_4}^{[2]}$.

We claim that $[h,\delta] = 1$. Clearly, $[h,\delta] \in G_{x_4,x_6}^{[2]}$. Thus, by Lemma 3.6 there is some $\gamma' \in G_{x_2,x_4}^{[2]}$ such that $[\gamma',\delta] = [h,\delta]$. Thus, $[\delta,\gamma' h^{-1}] = 1$, which is impossible unless $\gamma' = 1$. Hence $[h,\delta] = 1$. By Lemma 2.1 we thus have $h \in G_{x_2^{\delta}}^{[1]}$. Conjugation by δ^{-1} yields $h^{\delta^{-1}} = \alpha^{-1}\delta\beta^{-1}\alpha\beta\delta^{-1} = [\alpha,\beta\delta^{-1}] \in G_{x_2}^{[1]}$. Since $\alpha \in G_{x_2}^{[1]}$ we therefore must have $\alpha \in G_{x_2^{\delta\beta^{-1}}}^{[1]}$. But this is possible only if $x_3^{\delta\beta^{-1}} = x_3$. Thus, $\beta = \delta \in G_{x_6,x_8}^{[2]}$ and the lemma is proved. ∎

4.4 LEMMA: *If $U = G^{[2]}_{x_2,x_4}$, then the root action for x_0 is independent of the root.*

Proof: If $|U| = 2$, there is nothing to show. So we may assume that $|\Gamma_1(x_0)| \geq 4$. Let $x_4' \in \Gamma_1(x_3) \setminus \{x_2, x_4\}$. Let $U_1 = G^{[2]}_{x_2,x_4'}$, and let $\Omega = \Gamma_1(x_0) \setminus \{x_1\}$. By Lemma 3.7 both groups are abelian and regular on Ω.

We claim that $U_1|_\Omega = U|_\Omega$. By [1] 4.2A, it suffices to show that U_1 and U centralize each other, in particular then they centralize each other in their action on Ω. To see this let $\alpha \in U_1, \beta \in U$. Then by Lemma 3.6 there are $\gamma \in G^{[2]}_{x_0,x_2}$ and $\delta \in G^{[2]}_{x_4,x_6}$ such that $\beta = [\gamma, \delta]$. Then $[\alpha, \beta] = [\alpha, [\gamma, \delta]] = [\alpha, \gamma^{-1}\gamma^\delta] = 1$ by Lemma 3.6. Hence $U_1|_\Omega = U|_\Omega$.

Now let $(x_1, x_2'', x_3'', x_4'')$ be a simple path with $x_2'' \in \Gamma_1(x_1) \setminus \{x_0, x_2\}$, and let $U_2 = G^{[2]}_{x_2'',x_4''}$. Let $x_{-2} \in \Gamma_2(x_0) \setminus \Gamma_1(x_1)$. Now let $\gamma \in U_2, \beta \in U$ and let $h \in G^{[2]}_{x_{-2},x_0}$ with $x_3^{\gamma h} = x_3$. Then $\beta^{\gamma h}$ is an $(x_1, x_2, x_3, x_4^{\gamma h}, x_5^{\gamma h})$-elation. So $\beta^{\gamma h} \in U^{\gamma h}$ and by the previous step $U^{\gamma h}|_\Omega = U|_\Omega$. Since $\beta^{\gamma h}|_\Omega = \beta^\gamma|_\Omega$ and the situation is symmetric in U and U_2 we see that U_2 and U normalize each other on Ω.

Thus, $[U, U_2]|_\Omega \leq U|_\Omega \cap U_2|_\Omega$. If $U|_\Omega \cap U_2|_\Omega = 1$, then U and U_2 centralize each other as subgroups of Ω and, since they are abelian and regular, they must coincide by [1] Thm. 4.2A. Thus we must have $U|_\Omega \cap U_2|_\Omega \neq 1$.

Let $(x_4, x_3, x_2, x_1, x_2'', x_3'', x_4'', x_5'', x_6'', x_7'' = y_3, y_2, y_1, x_4)$ be a 12-cycle, and let $(y_3, y_4, y_5, y_6, y_7, x_0)$ be the path from y_3 to x_0. We can now use H. Van Maldeghem's argument (see [7] Prop. 4.6): Let $\alpha \in U \setminus \{1\}, \alpha_1 \in U_2 \setminus \{1\}$ with $\alpha\alpha_1^{-1} \in G^{[1]}_{x_0}$. Let $U_- = G^{[1]}_{y_3,y_4,y_5,y_6,y_7}$. Then $U_-^\alpha|_\Omega = U_-^{\alpha_1}|_\Omega$. There are $\beta \in U_-^\alpha, \beta_1 \in U_-^{\alpha_1}$ with $y_7^\beta = y_7^{\beta_1} = x_1$ and hence $U_-^\beta = U$ and $U_-^{\beta_1} = U_2$. But $\beta\beta_1^{-1} \in G^{[1]}_{x_0}$, and so $U|_\Omega = U_2|_\Omega$. ∎

4.5 LEMMA: *If $U = G^{[2]}_{x_2,x_4}$, then $U = G^{[2]}_{x_2,x_3,x_4}$.*

Proof: Let $U_1 = G^{[2]}_{x_0,x_2}$ and $y \in \Gamma_1(x_3) \setminus \{x_2, x_4\}$. By Lemma 4.4 and Corollary 3.4 there is some $g \in G^{[1]}_y$ with $x_2^g = x_4$ if $|\Gamma_1(x_0)| \geq 4$. If $|\Gamma_1(x_0)| = 3$, such an element g exists by Lemma 3.1. Let $U_2 = G^{[2]}_{x_4,x_0^g} = U_1^g$, so $U_2|_{\Gamma_1(y)} = U_1|_{\Gamma_1(y)}$. By Lemma 3.6 we have $[U_1, U_2] = U$. Since U and hence U_1 and U_2 are abelian, it follows that $U|_{\Gamma_1(y)} = 1$. Since $y \in \Gamma_1(x_3)$ was arbitrary, the claim follows. ∎

4.6 LEMMA: $[U, U] \leq G^{[2]}_{x_2,x_4}$.

Proof: Let $x_{-1} \in \Gamma_1(x_0) \setminus \{x_1\}$ and let $\beta \in G^{[1]}_{x_{-1},x_0,x_1,x_2,x_3}$. Then for $\alpha, \delta \in U$, we have $[\alpha, \delta^\beta] \in U \cap U^\beta$. But $[\alpha, \delta] \in U$ is also an elation, and since $\beta \in G^{[1]}_{x_0}$ we must have $[\alpha, \delta] = [\alpha, \delta^\beta]$. Thus, $[U, U] = [U, U^\beta] \leq U \cap U^\beta$ for any $\beta \in G^{[1]}_{x_0,x_1,x_2,x_3}$. As $G^{[1]}_{x_{-1},x_0,x_1,x_2,x_3}$ is transitive on $\Gamma_1(x_4) \setminus \{x_3\}$, we have $[U, U] \leq G^{[2]}_{x_4}$. Similarly, $[U, U] \leq G^{[2]}_{x_2}$. ∎

4.7 THEOREM: *If Γ is a half-Moufang generalized hexagon, then Γ is Moufang and the group generated by all elations of one type also contains all the elations of the other type, except in the case of $G_2(2)$ and the group generated by central elations.*

Proof: As always we suppose that the group $U = G^{[1]}_{x_1,\ldots,x_5}$ acts transitively on $\Gamma_1(x_0) \setminus \{x_1\}$.

By Lemma 4.3, either $U \cap G^{[2]}_{x_2,x_4} = 1$ or $U = G^{[2]}_{x_2,x_4}$. Note that in either case, U is abelian by Lemma 3.7 and Lemma 4.6, respectively.

We now consider the two cases separately:

$U = G^{[2]}_{x_2,x_4}$: By Lemma 4.5, $U = G^{[2]}_{x_2,x_3,x_4}$. Let $\alpha \in U$, and suppose $\alpha \notin G^{[3]}_{x_3}$. Either by Lemma 4.4 and Lemma 3.4 or, in case $|\Gamma_1(x_0)| = 3$, by Lemma 3.1, there is some $g \in G^{[1]}_{x_0,x_1,x_2}$ such that $1 \neq h = [g, \alpha] \in G^{[2]}_{x_2,x_3} \cap G^{[1]}_{x_0}$. As in the first part of the proof of Lemma 4.3, we now see that there is some $h' \in G^{[2]}_{x_1,x_3} \setminus \{1\}$. Now we are done by Lemma 3.1 and Proposition 3.5 applied to h'.

$G^{[2]}_{x_2,x_4} \cap U = 1$: Let $\alpha \in U$, and choose a 12-cycle

$$(x_0, x_1, \ldots, x_6, x_7, \ldots, x_{11}, x_0).$$

Let $\beta \in G^{[1]}_{x_3,x_4,x_5,x_6,x_7}$ such that $h = [\alpha, \beta] \neq 1$. We claim that $h \in G^{[2]}_{x_4}$. Let $y_1 \in \Gamma_1(x_4) \setminus \{x_3, x_5\}$, and let $(x_4, y_1, y_2, y_3, y_4, y_5, x_{10})$ be a path of length 6. Let $v \in G^{[1]}_{y_1,y_2,y_3,y_4,y_5}$ with $x_7^v = x_1$. Then $\beta^v \in U$, and since U is abelian, we have $[\alpha, \beta^v] = 1$. Since $v \in G^{[1]}_{y_1}$ we thus also have $[\alpha, \beta] \in G^{[1]}_{y_1}$. But y_1 was arbitrary, and so we have $h \in G^{[2]}_{x_4}$ as claimed. We next claim that $h \in G^{[2]}_{x_3}$. This is clear if $|\Gamma_1(x_0)| = 3$ since $h = [\alpha, \beta]$. So assume $|\Gamma_1(x_0)| > 3$. Let $z \in \Gamma_1(x_3)$ and let $w \in G^{[1]}_{x_7}$ with $x_2^w = z$ by Lemma 3.1. Then $[h, w] \in G^{[2]}_{x_4} \cap G^{[1]}_{x_6,x_7} = 1$ by assumption. By Lemma 2.1, $h \in G^{[1]}_z$ and hence $h \in G^{[2]}_{x_3}$. Similarly, $h \in G^{[2]}_{x_5}$. Again we are done by Lemma 3.1 and Proposition 3.5 applied to h.

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